Guaranteed integration on Lie groups

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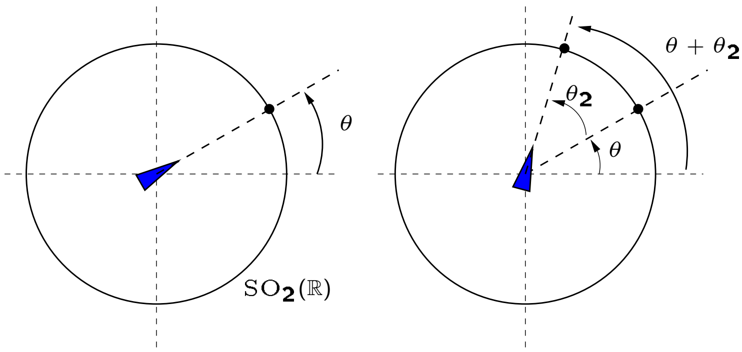
Outline

1. Lie group.
2. Differential equations on Lie group.
3. Guaranteed exponential operator.
4. Examples.

Lie groups

A Lie group is a *smooth manifold with a group structure*.

A Lie groups can represent simultaneously the state of a system (as an element of the manifold) and the transformation from one state to another (with the group operation).
E.g., $\mathrm{SO}(2)$ (2D rotations). An element can represent the heading of a car, or the rotation from a heading to another one.

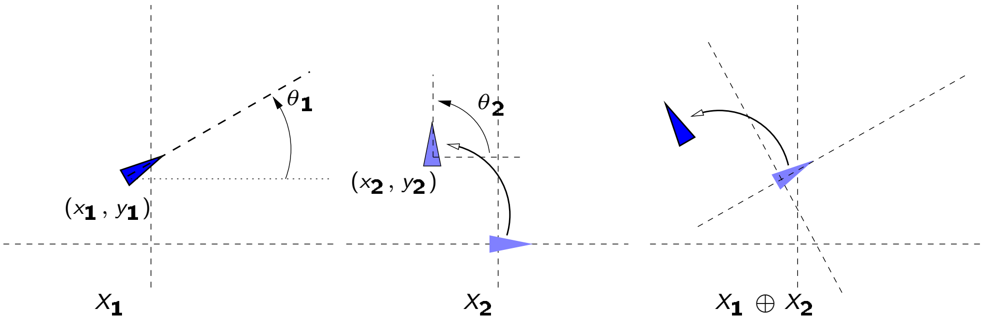


Matrix representation:

$$\theta \hookrightarrow R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
$$\theta_1 + \theta_2 \hookrightarrow R_{\theta_1} \cdot R_{\theta_2}$$

Example: SE(2)

$\mathrm{SE}(2)(\mathbb{R})$ combines rotations and tranlations on a plane.



Matrix representation:

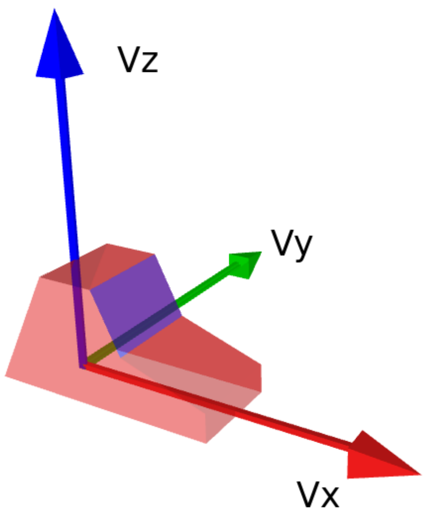
$$X \hookrightarrow M = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}$$

$$X_1 \oplus X_2 \hookrightarrow M_1 \cdot M_2 = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & x_1 + x_2\cos\theta_1 - y_2\sin\theta_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & y_1 + x_2\sin\theta_1 + y_2\cos\theta_1 \\ 0 & 0 & 1 \end{pmatrix}$$

As a transformation, x and y represents the translation *w.r.t. the current heading* ($x =$ forward, $y =$ on the side). Note that $M_1 M_2 \neq M_2 M_1$, thus $X_1 \oplus X_2 \neq X_2 \oplus X_1$.

Example: SO(3)

The group of 3D rotations (or the 3D orientation of a solid) can also be represented as a 3x3 matrix.



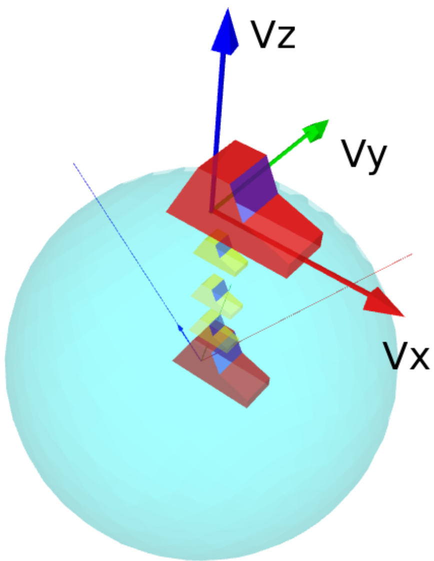
Matrix representation:

$$R \hookrightarrow M = (V_x \; V_y \; V_z)$$

M is a rotation matrix: $MM^T = Id$ (and $\det(M) = 1$). This constraint translates into 6 independant equalities on the components of V_x, V_y and V_z. Hence $\mathrm{SO}(3)$ is a manifold of dimension 3.

Graphical representation of SO(3)

We can also represent an element of SO_3 as the position and the heading of a car on a sphere: V_z is orthogonal to the surface of the sphere and V_x points forward.



For a sphere of radius r, the position of the car is rV_z.

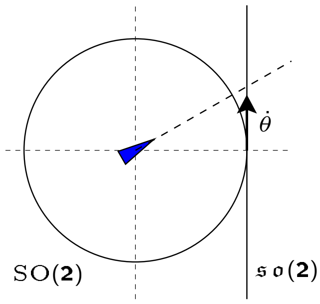
Lie algebra

As a group, a Lie group always as neutral element Id.
The *Lie algebra* $\mathfrak{g}$ associated to a Lie group G is the space tangent to G at the point Id. Its has the same dimension as G.

Example: for $\mathrm{SO}(2)$, with $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$,

time differentiation of R gives:

$$\dot{R} = \begin{pmatrix} -\dot{\theta}\sin\theta & -\dot{\theta}\cos\theta \\ \dot{\theta}\cos\theta & -\dot{\theta}\sin\theta \end{pmatrix}$$



which, near Id ($\theta = 0$):

$$\dot{R} = \begin{pmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{pmatrix} = \dot{\theta}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The Lie algebra is $\mathfrak{so}(2) = \mathbb{R}$, and $\dot{R}$ represents the matrix representation of $\dot{\theta}$. We denote this representation $\dot{\theta}^\wedge$.

Lie algebra for SE(2)

By extension, the Lie algebra describes the space tangent to G at any point of the manifold: let $A \in G$ and $\tau \in \mathfrak{g}$, then $A\tau^\wedge$ is tangent to G at A.

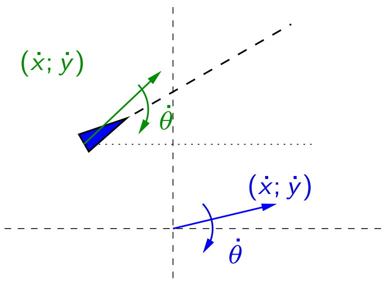
Example for SE(2)

An element of $\mathfrak{se}(2)$ is a vector $(\dot{x}\ \dot{y}\ \dot{\theta})$:

$$\dot{R} = \begin{pmatrix} 0 & -\dot{\theta} & \dot{x} \\ \dot{\theta} & 0 & \dot{y} \\ 0 & 0 & 0 \end{pmatrix} = (\dot{x}\ \dot{y}\ \dot{\theta})^\wedge$$

Let $R_0 = R(t_0)$. Then $\dot{R}(t_0) = R_0\,\tau^\wedge$ with $\tau^\wedge$ the differentiation of $t \mapsto R_0^{-1}R(t)$ at t_0. Then:

- $\dot{x}$ is the speed component relative to the current heading;
- $\dot{y}$ the speed component orthogonal to the current heading;
- $\dot{\theta}$ the angular speed.



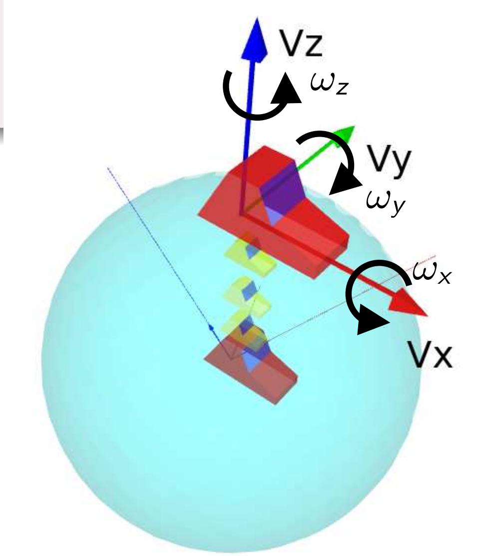
Lie algebra for SO(3)

Let $M \in \mathrm{SO}(3)$. From $MM^T = \mathrm{Id}$, we get $\dot{M}M^T + M\dot{M}^T = 0$. Thus $\dot{M}M^T$ is skew-symmetric, and when $M = \mathrm{Id}$:

$\mathfrak{so}(3)$

The Lie algebra $\mathfrak{so}3$ is the space of 3x3 skew-symmetric matrices.

$$\dot{M} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$



$\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ is the vector of angular velocity in local coordinates.
On the sphere:

- $\omega_y > 0$ represents the car moving forward;
- $\omega_z > 0$ represents the car turning left;
- $\omega_x > 0$ represents the car "sliding" to the right.

Differential equation

A differential equation on a Lie group G (with the Lie algebra $\mathfrak{g}$) is of the form:

$$\dot{R}(t) = f(R, t) \qquad R(t_0) = R_0$$

with $f : G \times \mathbb{R} \to \mathfrak{g}$.
Hence, at each instant t:

$$\begin{aligned} R(t + dt) &= R(t) + R(t)\, f(R(t), t)^{\wedge} dt \\ &= R(t)\, (\mathrm{Id} + f(R(t), t)^{\wedge} dt) \end{aligned}$$

Constant case: $\dot{R}(t) = \boldsymbol{v}$

When $f(R, t) = \boldsymbol{v}$ is constant, we have:

$$R(1) = R(0) \lim_{dt \to 0} \prod_{1}^{1/dt} (\mathrm{Id} + \boldsymbol{v}^{\wedge} dt)$$

Then:

$$R(1) = R(0) \sum_{n \geq 0} \frac{(\boldsymbol{v}^{\wedge})^n}{n!} = R(0) e^{\boldsymbol{v}^{\wedge}}$$

and more generally,

$$R(t) = R(t_0) e^{(t - t_0)\boldsymbol{v}^{\wedge}}$$

Case $\dot{R}(t) = f(t)$

Two equivalent "product integral" equations:

$$
\begin{aligned}
R(t) &= R(t_0)\ (\mathrm{Id} + f(t_0 + d\tau)d\tau)(\mathrm{Id} + f(t_0 + 2d\tau)d\tau)\ldots \\
&= R(t_0)\ (1 + f(\tau)d\tau) \prod_0^t \\
\text{or } R(t) &= R(t_0)\ e^{f(\tau)d\tau} \prod_0^t
\end{aligned}
$$

- these are "right-product integrals" (symbol $\prod_0^t$ at the *right* of the expression);
- the second expression is uncommon, but its discrete approximation converges faster than the first one.

A classical (not guaranteed) approximation of $R(t)$ with N steps is therefore (with $\delta t = (t - t_0)/N$):

$$R(t) = R(t_0) e^{\frac{f(t_0)}{N}} e^{\frac{f(t_0+\delta t)}{N}} \ldots e^{\frac{f(t)}{N}}$$

Example

In SE_2, let's consider, with $t \in [0,1]$:

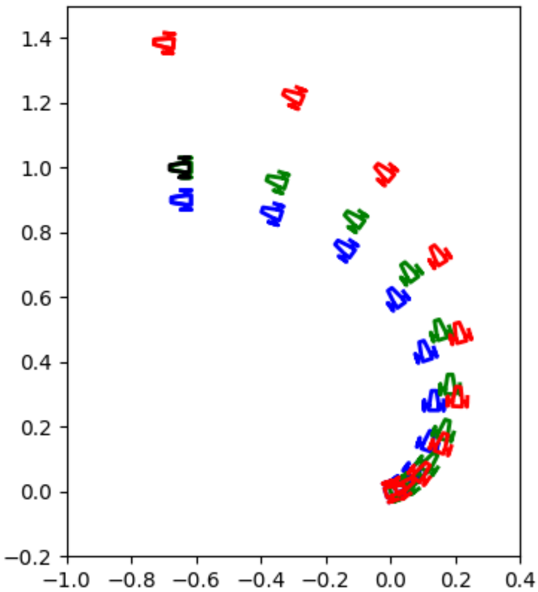
$$f(t) = \begin{pmatrix} 0 & -\pi & \pi t \\ \pi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

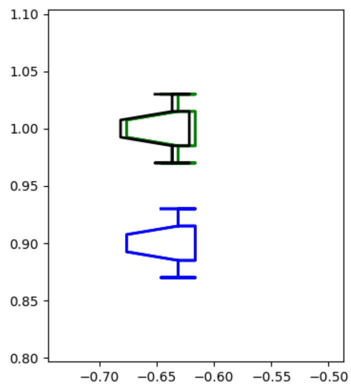
and $R(0) = \mathrm{Id}$.

$$R(1) = e^{f(t)dt} \prod_0^1 = \begin{pmatrix} -1 & 0 & -2/\pi \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Red: "basic" Euler ($\mathrm{Id} + f(t)\delta t$) ; blue : "exponential" Euler ($e^{f(t)\delta t}$) ; green : with midpoint ($e^{f(t+\delta t/2)\delta t}$). Black : exact final position.

Despite f being affine, using the midpoint does not give an exact result.





Issue with product of exponential

Classical result

Let $f : [0,1] \to \mathbb{R}^n$ continuous, such that $f([0,1]) \subseteq [B]$. Then:

$$\int_0^1 f(t)dt \in [B]$$

This result enables to replace f by a (convex) over-approximation of its image to get a guaranteed integration.
However, in the general case:

$$e^{f(t)dt} \prod_{t_0}^{t_1} \neq e^{\int_0^1 f(t)dt}$$

and

$$e^{f(t)dt} \prod_{t_0}^{t_1} \notin \{e^v \mid v \in [B]\}$$

Example

Still in SE_2:

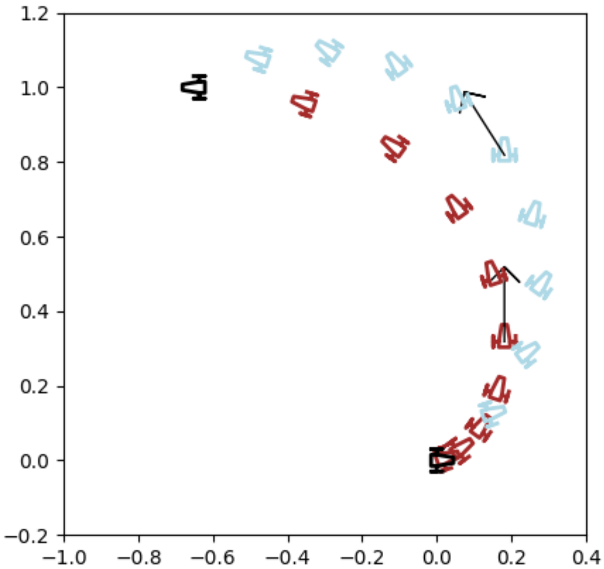
$$f(t) = \begin{pmatrix} 0 & -\pi & \pi t \\ \pi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We recall:

$$e^{f(t)dt} \prod_0^1 = \begin{pmatrix} -1 & 0 & -2/\pi \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

An example of v such that $e^v = e^{f(t)dt} \prod_0^1$ can be (light blue):

$$v = \begin{pmatrix} 0 & -\pi & \pi/2 \\ \pi & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



But v is not in $[f(t)]$.

Guaranteed exponential operator

The issue stems from the fact that in general, $e^A e^B \neq e^{A+B}$ (thus from $AB \neq BA$).

Exponential operator for integration

Let $[B]$ a box. We define $\exp([B])$ (not equal to $e^{[B]}$) as:

$$\exp([B]) = \{e^{f(t)dt} \prod_0^1 \mid f : [0,1] \to [B]\}$$

Note: in this definition, the side of the product does not matter:

$$\exp([B]) = \{\prod_0^1 e^{f(t)dt} \mid \ldots\}$$

Example

With

$$[B] = \begin{pmatrix} 0 & \pi & [0,\pi] \\ -\pi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

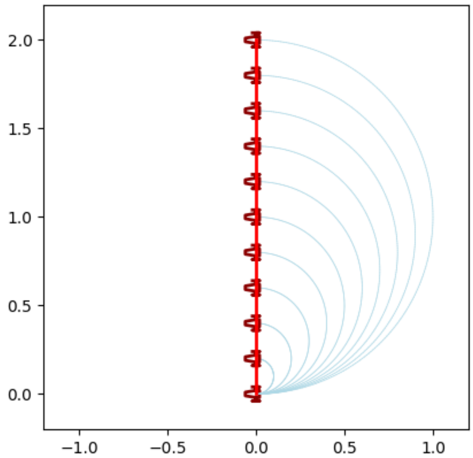
$e^{[B]}$ is in red, $\exp([B])$ in blue.

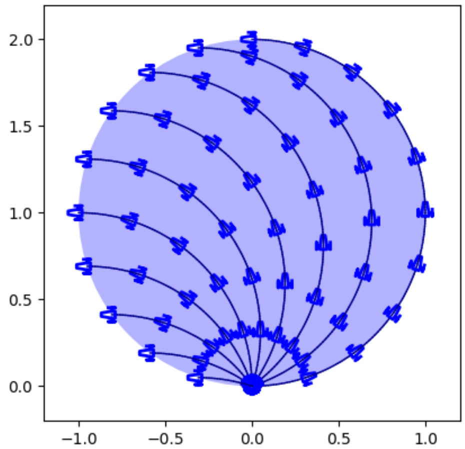
$$e^{[B]} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & [0,2] \\ 0 & 0 & 1 \end{pmatrix}$$

$$\exp([B]) = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

s.t.

$$x^2 + (y-1)^2 \leq 1$$





Guaranteed exponential operator

Main result:

$$\exp([B]) \subseteq Id + [B] + \frac{[[B][B]]}{2!} + \frac{[[B][B][B]]}{6!} + \ldots$$

Let's precise that:

- $[B][B] = \{B_1 B_2 \;\mid\; B_1 \in [B], B_2 \in [B]\} \;\neq\; [B]^2$
- we must use the convex hull (box) after the computation of products;
- factorisations (Horner's scheme) are not possible.

Justification, error term

Justification: let's consider $e^{U/2}e^{V/2}$ with $U, V \in [B]$.

$$e^{U/2}e^{V/2} = Id + \left(\frac{U}{2} + \frac{V}{2}\right) + \frac{1}{2}\left(\frac{U^2}{4} + \frac{UV}{2} + \frac{V^2}{4}\right) + \dots$$

Second-order term $(\frac{U^2}{4} + \frac{UV}{2} + \frac{V^2}{4})$ is in the convex hull of U^2, UV and V^2, thus in the convex hull of $[B][B]$. Similar result can be shown on higher-order terms.

The error term is similar to the error term for $e^{[B]}$:

$$\exp([B]) \subseteq \sum_{k=0}^{N} \dots \ + E_N$$

with

$$E_N = \frac{||[B]^{N+1}||_\infty^{N+1}}{(N+1)!(1 - \frac{||B||_\infty}{N+2})} \mathrm{mat}_{n,n}([-1,1])$$

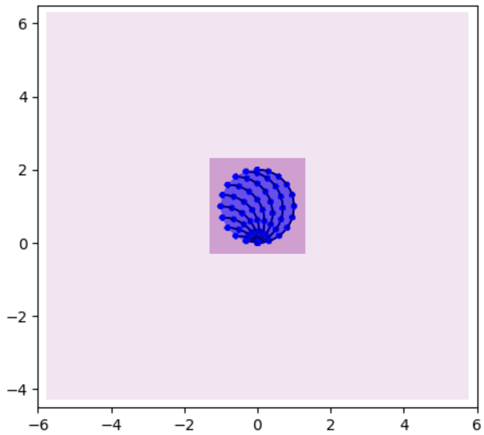
Scaling and squaring

Quite easily:

$$\exp([B]) \subseteq \exp(\frac{1}{2}[B]) \cdot \exp(\frac{1}{2}[B])$$

This equation can be used when $||B||_\infty$ is too large.

Light purple: 30 iterations, without scaling and squaring.
Dark purple: using scaling and squaring.
Since the matrix is quite large, the precision is limited.



State representation and contraction

Note: from now on, we consider $[B]$ to have a small radius.
Given $[B] = C + [R]$ (with $[R]$ centered around 0), we express $\exp([B])$ as:

$$\exp([B]) \subseteq [M][e^C]$$

where $[e^C]$ is quasi-punctual. Then $[M]$ is a small box around Id.

- this can be done by computing $[M] = [e^C]^{-1}\exp([B])$ (other approaches did not give any improvement), and $[e^C]^{-1} = [e^C]^T$ in $\mathrm{SO}(3)$;
- since $[M]$ represents a set of Lie group elements, we can contract it accordingly (contraction around Id works fairly well).

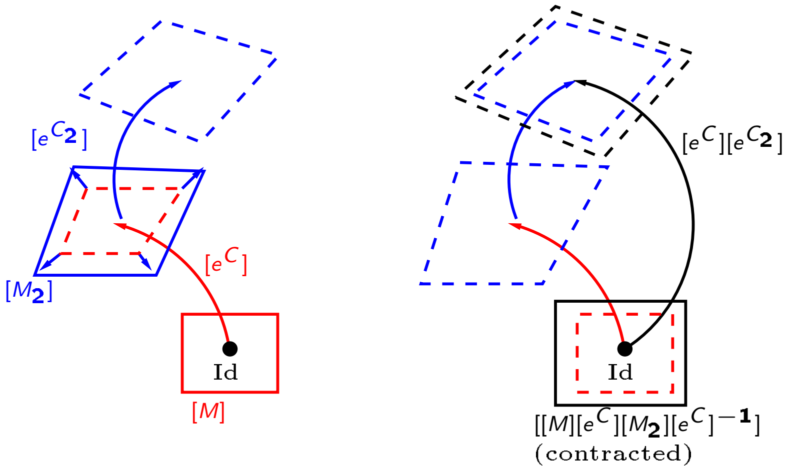
Several-steps guaranteed integration

We keep this representation $[M][e^C]$ with $[e^C]$ punctual, and $[M]$ "small" around Id during the whole integration:
At each step, we approximate $\exp([B])$ as $[M_2][e^{C_2}]$. Then we use:

$$[M][e^C][M_2][e^{C_2}] \subseteq ([M][e^C][M_2][e^C]^{-1})[e^C][e^{C_2}]$$

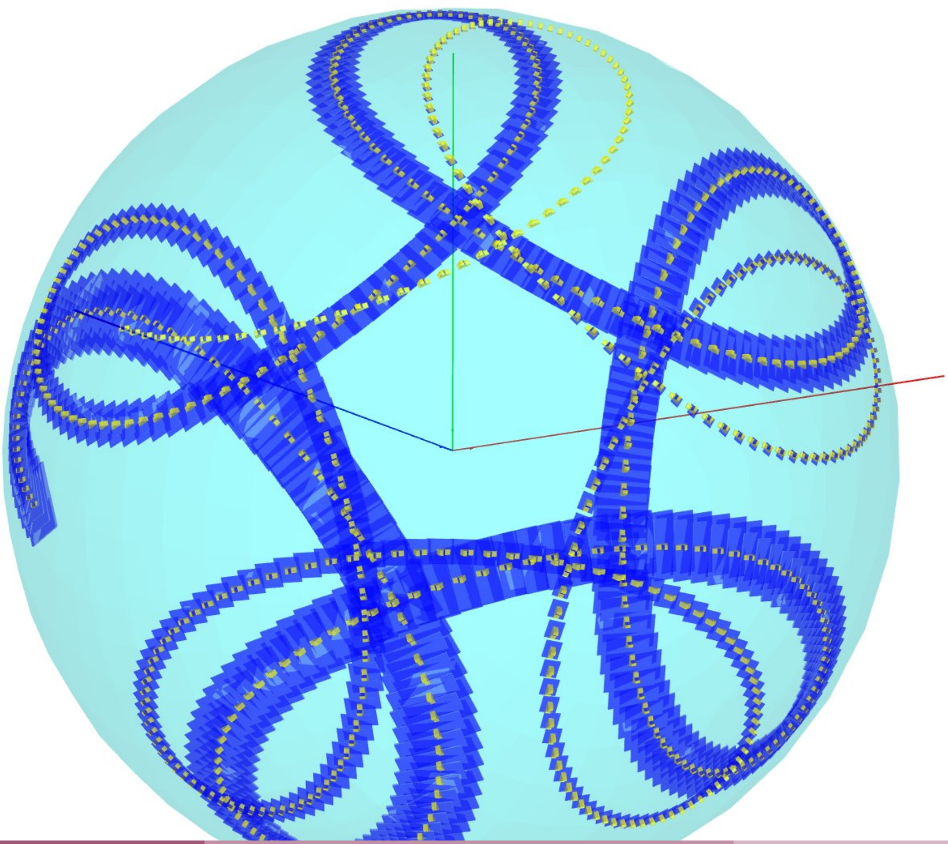
Thus $[M'] = [[M]([e^C][M_2][e^C]^{-1})]$ (contracted) and $[e^{C'}] = [e^C][e^{C_2}]$.

- avoid the cumulated uncertainties ($[M]$) to "explode" by rotation;
- works better with $[M] - \mathrm{Id}$ (similarly, $[e^C][M_2][e^C]^{-1}$ is computed as $\mathrm{Id} + [e^C]([M_2] - \mathrm{Id})[e^C]^{-1}$).



Application

"Guaranteed" integration of gyro data: for each temporal step $[t_0, t_1]$, we consider f varying inside a box.



Discussion

We can also describe the problem as a 9-dimensional differential equation/inclusion ($X \in \mathbb{R}^9$):

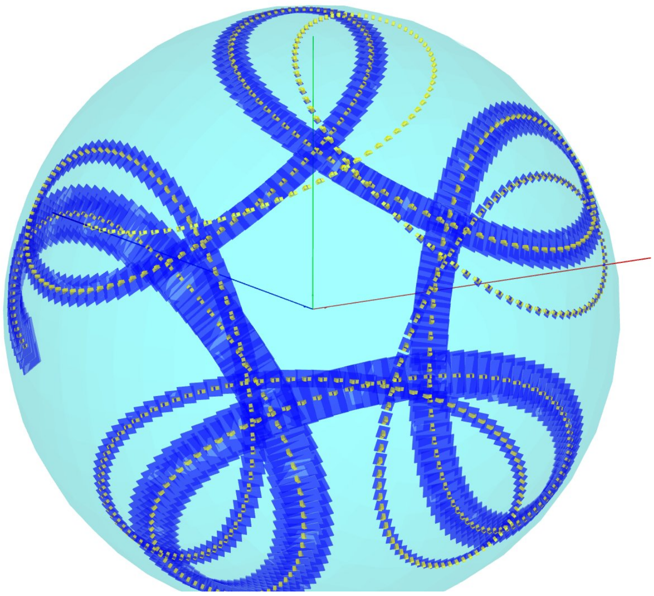
$$\dot{X} = A(t) \cdot X$$

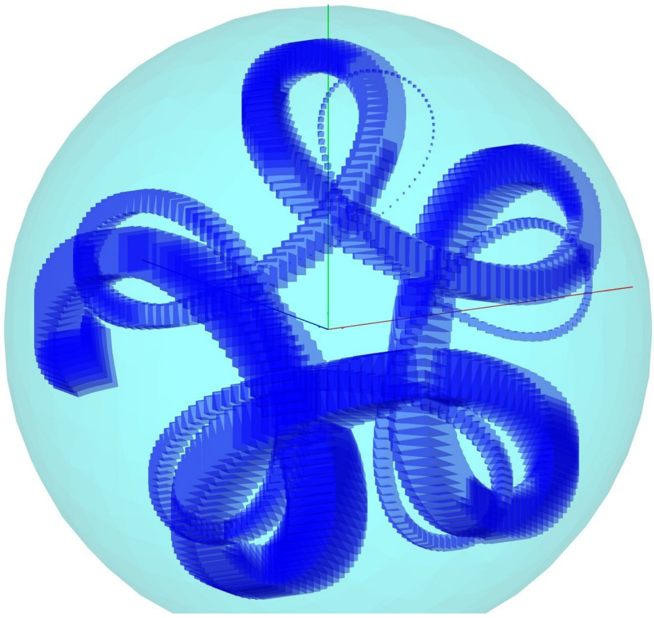
Then A is a 9x9 matrix. A common approach would be to approximate $A(t) \cdot X$ (on a small step) by $A_m \cdot X + [B]$ and safely approximate the evolution of X using e^{A_m} (here, A_m is punctual):

$$X(t+\delta t) - X_m \in e^{\delta t A_m}(X(t) - X_m) + \int_0^{\delta t} e^{(\delta t - \tau)A_m}[B] d\tau$$

The representation of the reachable states requires to use abstract domains, at least some kind of parallelotops ($[X] = \{Qx \mid x \in [v]\}$. Compared with the previous representation $[X] = [M][e^C]$, we can associate the quasi-punctual Q (81 elements) with $[e^C]$ (9 elements), and $[M]$ with $[v]$. The approach is slower (more dimensions) and lacks the specific contractors used for $\mathrm{SO}(3)$.

Discussion (2)





Conclusion

1. "Simple" guaranteed integration over Lie Group.
2. Enable to use the specific properties on the Group (contractor).
3. Extension to different groups.
4. How to handle more complex equations?